M1220-2 **Midterm 3** **Spring 2005**

Midterm 3 Scores (out of 100): n = 130; mean = 68

25th percentile = 58; median (50th percentile) = 69; 75th percentile = 80

Note: Average of Two Best Midterm Scores (out of 100) for the class: n = 140; mean = 73

25th percentile = 63; median (50th percentile) = 78; 75th percentile = 87

1. (10 pts) In the following determine whether the sequence $\{a_n\}$ converges or diverges. If the sequence converges, give the limit of the sequence. If the sequence diverges, briefly state why.

i)

$$a_n = \left(\frac{e}{-10\pi}\right)^n$$

This sequence is of the form $\{r^n\}$ with $|r| < 1$ so this sequence converges to zero.

ii)

$$a_n = (-1)^n \frac{5n^2+1}{2n^2-1}$$

The terms in this sequence alternate between positive and negative values. Since $\lim_{n\to+\infty} \frac{5n^2+1}{2n^2-1} = 5/2$, as n approaches infinity, the positive terms will approach 5/2 while the negative terms will approach -5/2. As a result, this sequence diverges or does not have a limit.

2. (10 pts) **Give an example of each of the following:**

i. A convergent geometric series

Any series of the form :$\sum_0^{+\infty} ar^n$ where a is a constant and $|r| < 1$.

ii. A divergent p- series

Any series of the form :$\sum_1^{+\infty} \frac{1}{n^p}$ where $p \leq 1$.

3.(30 pts) Determine whether the following series **diverge, converge conditionally, or converge absolutely**. Justify your answer completely.

i.

$$\sum_{2}^{+\infty}(-1)^n \frac{1}{n \ln n}$$

First we consider the series of absolute values of the terms. We compare the series, $\sum \frac{1}{n(\ln n)}$ to the integral, $\int_2^{+\infty} \frac{1}{x \ln x} dx$, which letting $u = \ln x$ becomes $\int_{\ln 2}^{+\infty} \frac{du}{u}$. Since this last integral diverges, the series, $\sum \frac{1}{n(\ln n)}$, diverges and we know the given series does not converge absolutely.

However, the given series is an alternating series with terms that are decreasing and that approach zero as n approaches infinity, so the series, $\sum_2^{+\infty}(-1)^n \frac{1}{n \ln n}$, converges. For this reason, the series, $\sum_2^{+\infty}(-1)^n \frac{1}{n \ln n}$, is said to be conditionally convergent.

ii.

$$\sum_{1}^{+\infty}(-1)^n \frac{\cos n}{n^3}$$

Note that for all n, $0 \leq |(-1)^n \frac{\cos n}{n^3}| \leq \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ is a p-series with $p > 1$. this series converges. Then by the Comparison Test, the series, $\sum_1^{+\infty}(-1)^n \frac{\cos n}{n^3}$, is absolutely convergent.

iii.

$$\sum_{1}^{+\infty}(-1)^n \frac{n}{5n-3}$$

Note that $\lim_{n \to +\infty} \frac{n}{5n-3} = 1/5$. This together with the fact that the terms in the series alternate between positive and negative values, means that the limit of the terms in the series does not exist. Then, since $\lim_{n \to +\infty} a_n \neq 0$, by the Nth term Divergence Test, the series diverges.

4. (24 pts) **Power Series:** Find the convergence set for the following two power series.

i.

$$\sum_{1}^{+\infty} \frac{n^2x^n}{n!}$$

Note that: $\lim_{n\to+\infty} |\frac{(n+1)^2x^{n+1}}{(n+1)!} \frac{n!}{n^2x^n}| = \lim_{n\to+\infty} |(\frac{n+1}{n})^2 \frac{1}{n+1}|x|$
$= |x| \lim_{n\to+\infty} (\frac{n+1}{n})^2 \frac{1}{n+1} = 0$. So, by the Absolute Ratio Test, this series converges for all real numbers. That is it's convergence set is $(-\infty, +\infty)$.

ii.

$$\sum_{1}^{+\infty} \frac{(x-3)^n}{\sqrt{n}}$$

Note that: $\lim_{n\to+\infty} |\frac{x-3)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(x-3)^n}| = |x-3| \lim_{n\to+\infty} \sqrt{\frac{n}{n+1}} = |x-3|$. So this series will converge on the interval where $|x-3| < 1$, or $2 < x < 4$ by the Absolute Ratio Test. Since the $|x-3| = 1$ when $x = 2, 4$ the Absolute Ratio Test is inclusive at these values of x and we must test for convergence at these endpoints separately. At $x = 2$, the series becomes $\sum \frac{(-1)^n}{\sqrt{n}}$ which is a convergent series by the Alternating Series Test since the terms in the series decrease and approach zero as n approaches infinity. At $x = 4$, the series becomes $\sim \frac{1}{\sqrt{n}}$ which is a divergent p-series. Then the convergence set for this series is the interval [2, 4).

5. (20 pts) **Taylor Series:**
i. (8 pts) Find the Taylor Polynomial of order 3 based at $\frac{\pi}{4}$ for $f(x) = \sin(x)$.

Note that:

$f(x) = \sin x$	$f^{(0)}(\pi/4) = \frac{1}{\sqrt{2}}$
$f^{'}(x) = \cos x$	$f^{(0)}(\pi/4) = \frac{1}{\sqrt{2}}$
$f^{''}(x) = -\sin x$	$f^{(0)}(\pi/4) = -\frac{1}{\sqrt{2}}$
$f^{'''}(x) = -\cos x$	$f^{(0)}(\pi/4) = -\frac{1}{\sqrt{2}}$.

So, the 3rd order Taylor Polynomial for $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{\sqrt{2}}(x - \pi/4)^2 - \frac{1}{\sqrt{2}}(x - \pi/4)^3$.

ii. (6 pts) Use the Maclaurin Series for $f(x) = e^x$ to find the Maclaurin Series for $g(x) = e^{x^3}$. On what interval will this series converge? Describe the series using summation notation or give at least 5 terms in the series to show the pattern of the series.

Since the Maclaurin series for e^t is: $\sum_{n=0}^{+\infty} \frac{t^n}{n!} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...$, which converges for all real x, if we let $t = x^3$, then the Maclaurin series for e^{x^3} is $\sum_{n=0}^{+\infty} \frac{x^{3n}}{n!} = 1 + \frac{x^3}{1!} + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + ...$ This series converges on the interval $(-\infty, +\infty)$.

iii. (6 pts) Use your answer in ii. above to find the Maclaurin Series for the function $h(x) = 3x^2e^{x^3}$. Describe the series using summation notation or give at least 5 terms in the series to show the pattern of the series.

Since $h(x) = 3x^2e^{x^3}$ is the derivative of the function $f(x) = e^{x^3}$, we can differentiate term by term to find that the Maclaurin series for $3x^2e^{x^3}$ is $3x^2 + \frac{6}{2!}x^5 \frac{9}{3!}x^8 + \frac{12}{4!}x^{11} + \frac{15}{25!}x^{14} + ...$ We obtain the same result by multiplying the series for e^{x^3} by $3x^2$.

6. (6 pts) Determine whether the following statements are True or False. If a statement is False, give a counterexample that shows why the statement is False.

i. If $\lim_{n\to+\infty} a_n = 0$, then the infinite series $\sum a_n$ converges.

This statement is false. Any p- series with$0 < p < 1$ is a good counterexample since $\lim_{n\to+\infty} \frac{1}{n^p} = 0$, yet the series will diverge.

ii. If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

This statement is true.

iii. If $\sum a_n, \sum b_n$ both diverge, then $\sum(a_nb_n)$ will diverge.

This statement is false. If $a_n = 1/n, b_n = 1/n$, then $\sum a_n, \sum b_n$ both diverge, yet $\sum(a_nb_n) = \sum 1/n^2$, a convergent series.